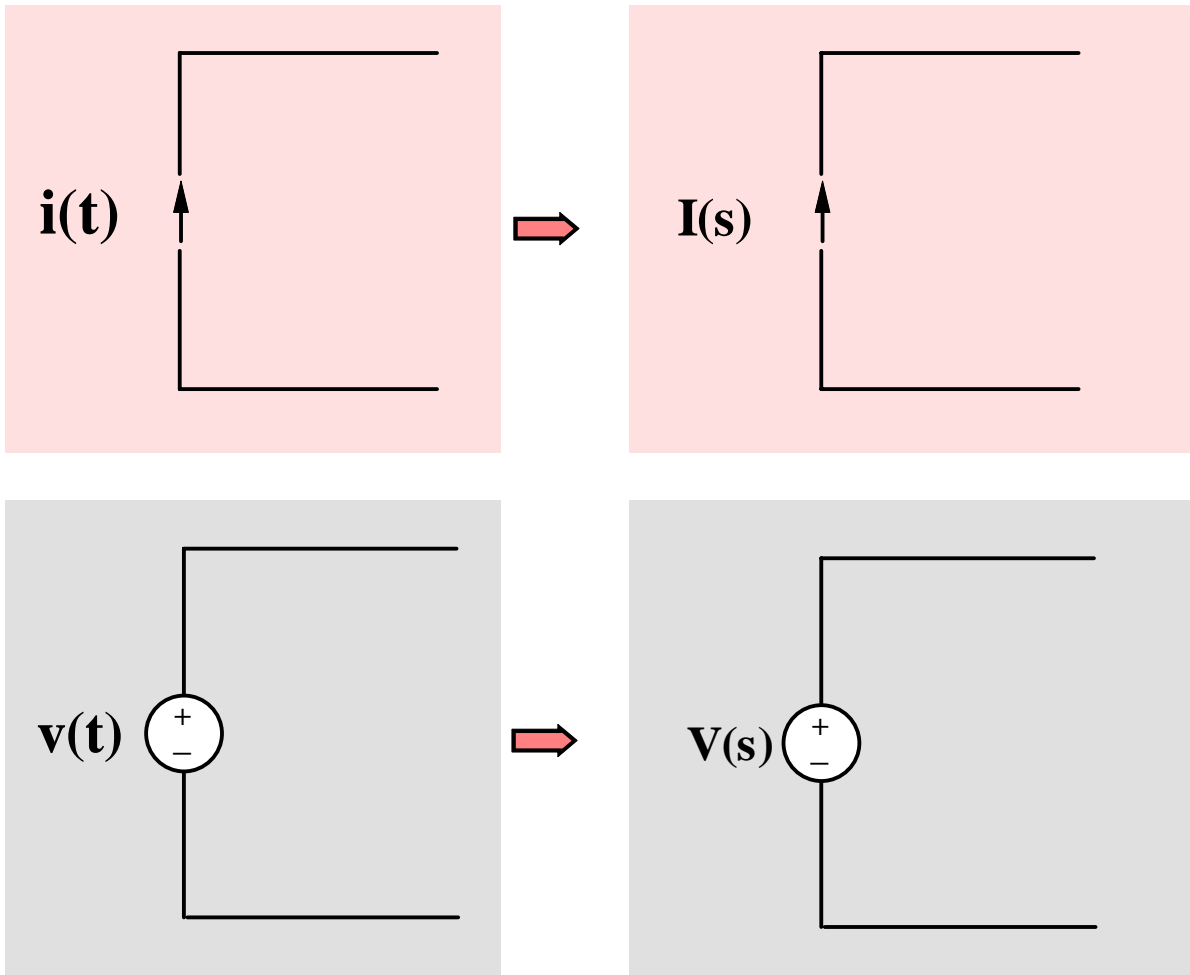


Laplace Circuit Analysis

Circuit Analysis in the “S” Domain

Laplace Circuit Analysis

Circuit Elements in the “S” Domain



Laplace Circuit Analysis

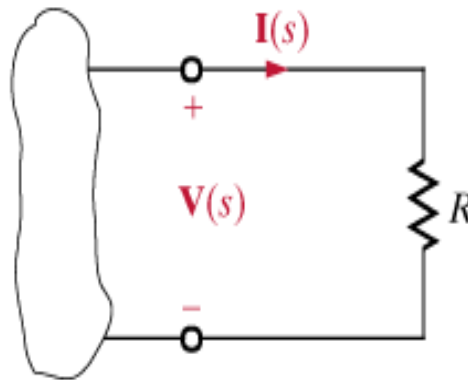
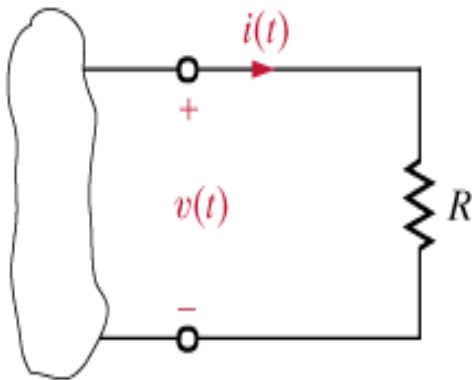
Circuit Element Modeling

The method used so far follows the steps:

1. Write the differential equation model
2. Use Laplace transform to convert the model to an algebraic form

1.0 Resistance

Resistor

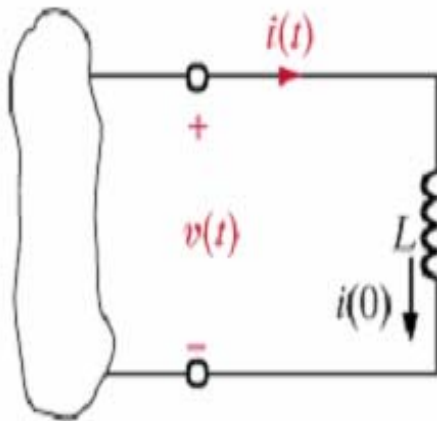


$$v(t) = Ri(t) \Rightarrow V(s) = RI(s)$$

(a)

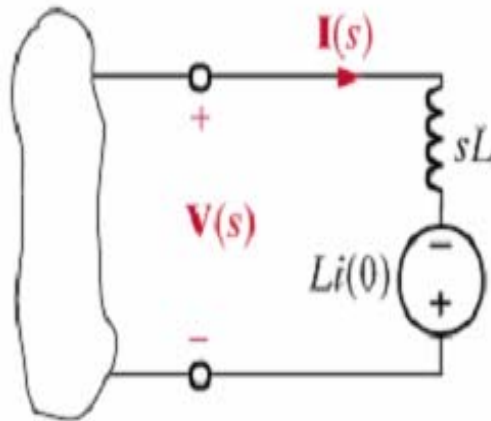
Laplace Circuit Analysis

2.0 Inductor



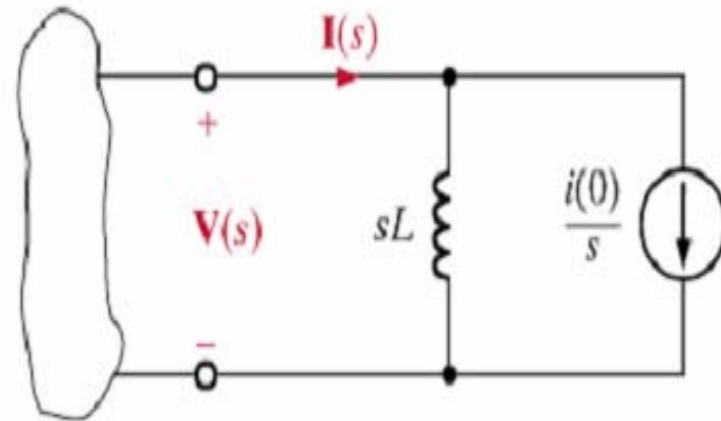
$$v(t) = L \frac{di}{dt}(t)$$

\Rightarrow



$$V(s) = LsI(s) - Li(0)$$

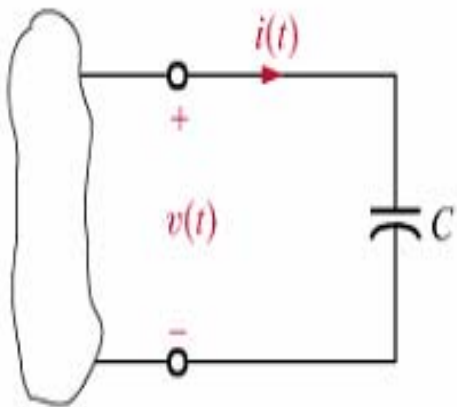
\Rightarrow



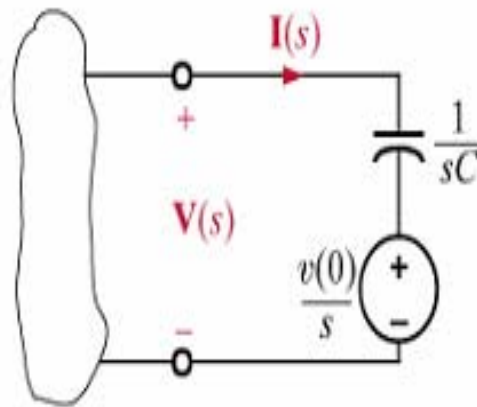
$$I(s) = \frac{V(s)}{Ls} + \frac{i(0)}{s}$$

Laplace Circuit Analysis

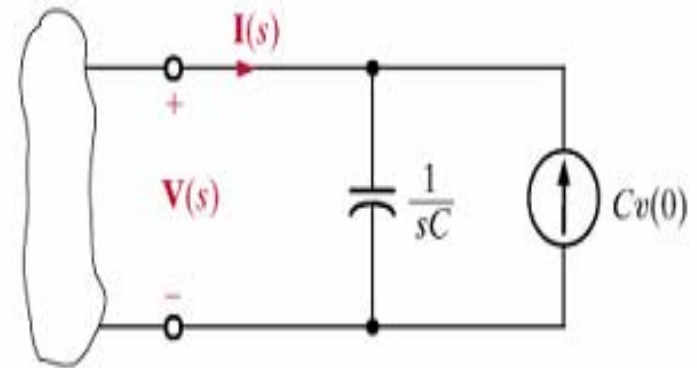
3.0 Capacitor



$$v_c(t) = \frac{1}{C} \int_0^t i(t) dt + v_c(0)$$



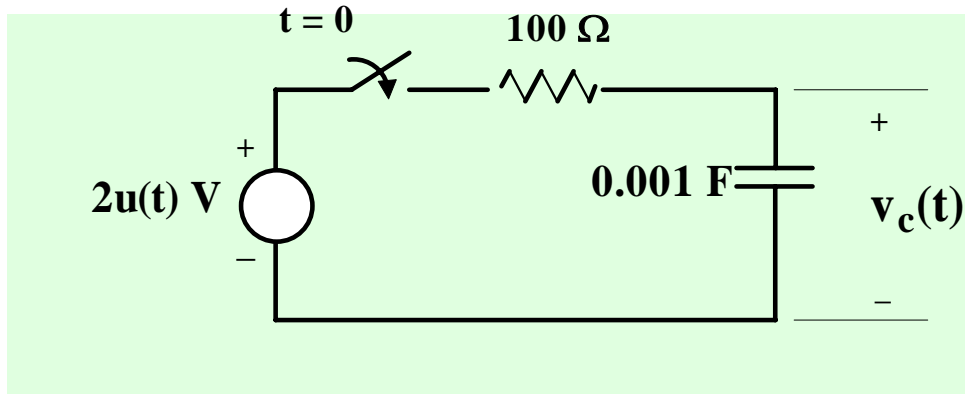
$$V(s) = \frac{1}{Cs} I(s) + \frac{v(0)}{s}$$



$$I(s) = CsV(s) - Cv(0)$$

Example

Given the circuit below. Assume $v_c(0) = -4 \text{ V}$.
Use Laplace to find $v_c(t)$.

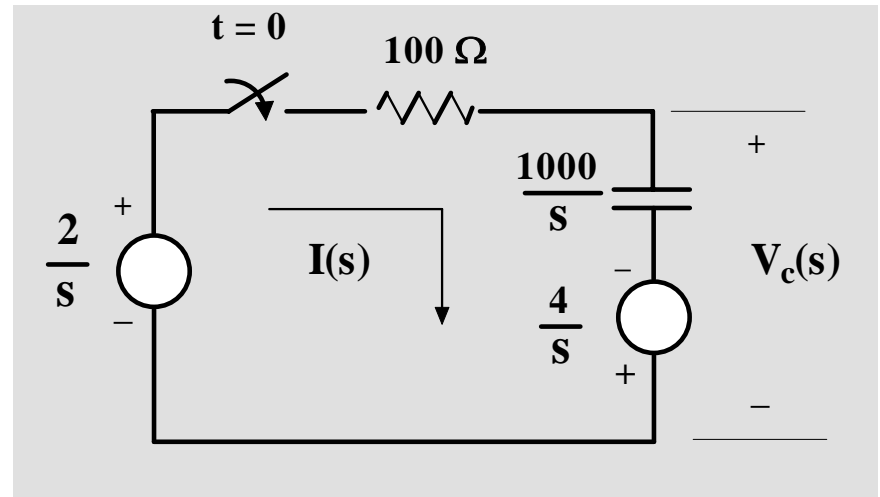


Laplace circuit:

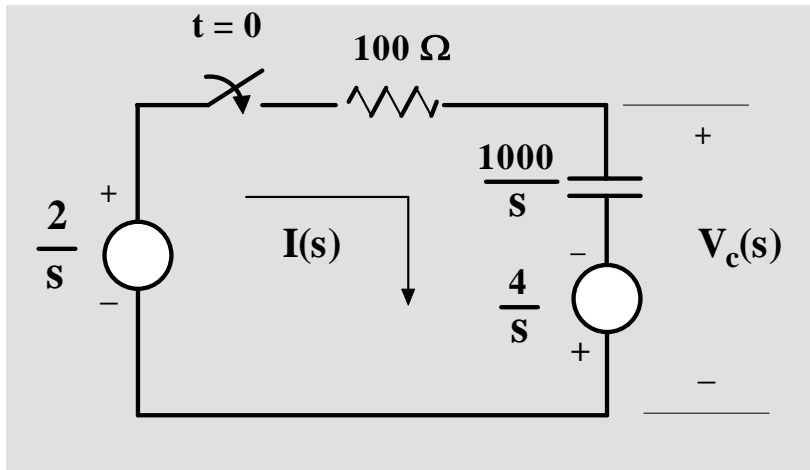
Calculate the current

$$\frac{2}{s} + \frac{4}{s} = I(s) \left[100 + \frac{1000}{s} \right]$$

$$100I(s) = \frac{6}{s + 10}$$



Calculate the voltage



①

$$\frac{2}{s} - 100I(s) - V_c(s) = 0$$

$$\frac{2}{s} - \frac{6}{s+10} = V_c(s)$$

②

$$V_c(s) = \frac{-4s+20}{s(s+10)} = \frac{A}{s} + \frac{B}{s+10}$$

$$V_c(s) = \frac{2}{s} - \frac{6}{s+10}$$

$$v(t) = [2 - 6e^{-10t}]u(t)$$

Check the boundary conditions

③

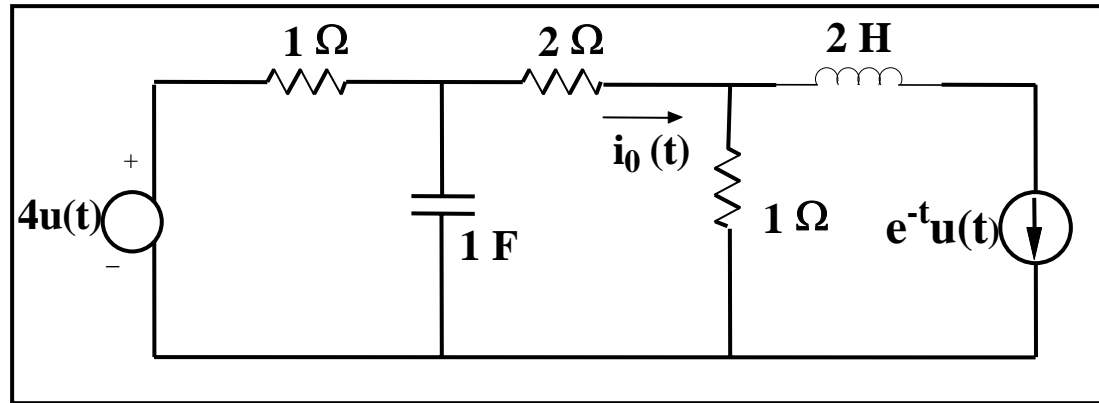
$$v_c(0) = -4 \text{ V}$$



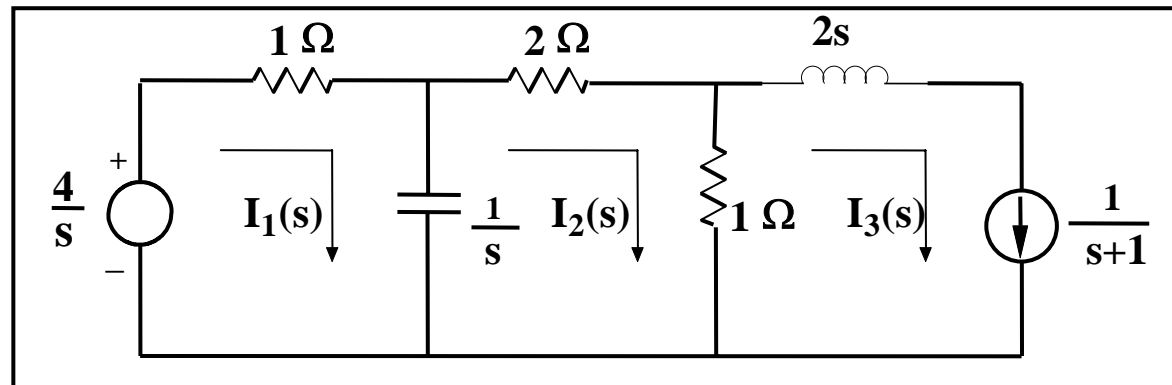
Example

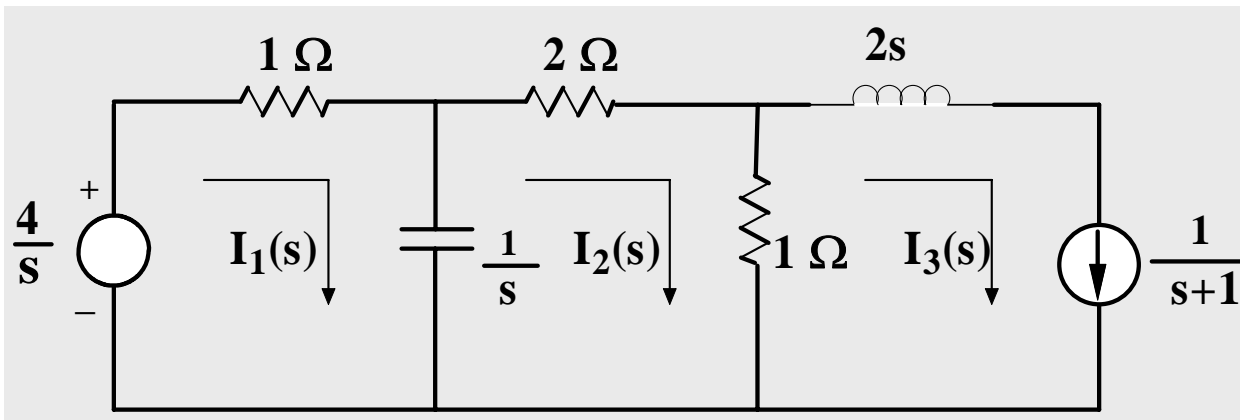
Find $i_0(t)$ using mesh current method

Time Domain



Laplace





Mesh 1

$$\frac{(s+1)}{s}I_1(s) - \frac{I_2(s)}{s} = \frac{4}{s}$$

$$(s+1)I_1(s) - I_2(s) = 4$$

Mesh 3

$$I_3(s) = \frac{1}{s+1}$$

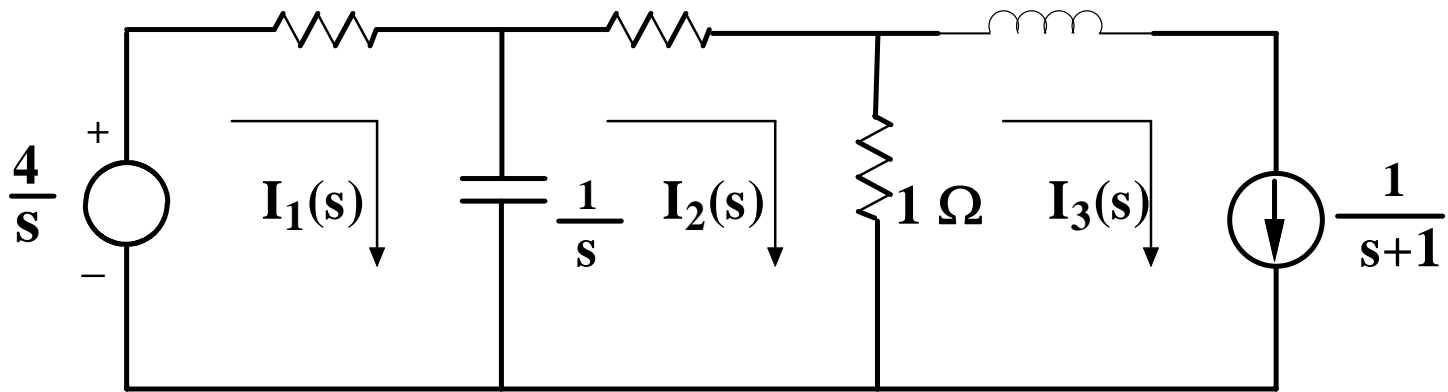
Mesh 2

$$-\frac{1}{s}I_1(s) + \frac{3s+1}{s}I_2(s) - I_3(s) = 0$$

$$-\frac{1}{s}I_1(s) + \frac{3s+1}{s}I_2(s) - \frac{1}{s+1} = 0$$

$$-I_1(s) + (3s+1)I_2(s) = \frac{s}{s+1}$$

$$-(s+1)I_1(s) + (s+1)(3s+1)I_2(s) = s$$



$$s(3s + 4)I_2(s) = s + 4$$

$$I_2(s) = \frac{\left(\frac{1}{3}\right)(s + 4)}{s(s + \frac{4}{3})} = \frac{1}{s} - \frac{\frac{2}{3}}{s + \frac{4}{3}}$$

$$i_2(t) = \left[1 - \frac{2}{3}e^{-\frac{4}{3}t}\right]u(t)$$

lo

Analysis in the s-domain has established that the Laplace transform of the output voltage is

$$V_o(s) = \frac{12(s+3)}{s(s^2+4s+5)} \quad s^2+4s+5 = (s+2-j1)(s+2+j1) = (s+2)^2+1$$

$$V_o(s) = \frac{12(s+3)}{s(s+2-j1)(s+2+j1)} = \frac{K_o}{s} + \frac{K_1}{s+2-j1} + \frac{K_1^*}{s+2+j1}$$

$$K_o = sV_o(s)|_{s=0} = 36/5$$

$$\frac{K_1}{(s+\alpha-j\beta)} + \frac{K_1^*}{(s+\alpha+j\beta)} \leftrightarrow 2|K_1|e^{-\alpha t} \cos(\beta t + \angle K_1)u(t)$$

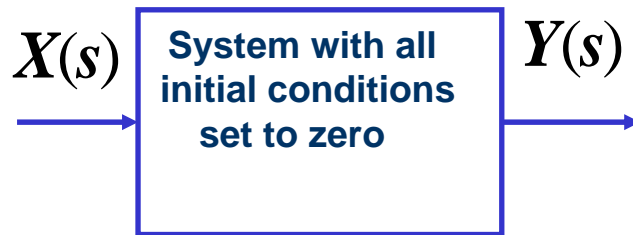
$$K_1 = (s+2-j1)V_o(s)|_{s=-2+j1} = \frac{12(1+j1)}{(-2+j1)(j2)} = \frac{12\sqrt{2}\angle 45^\circ}{\sqrt{5}\angle 153.43^\circ (2\angle 90^\circ)}$$

$$= 3.79\angle -198.43^\circ = 3.79\angle 161.57^\circ$$

$$v_o(t) = \left(\frac{36}{5} + 7.59e^{-2t} \cos(t + 161.57^\circ) \right) u(t)$$

TRANSFER FUNCTION

The Transfer Function (T.F.) is the ratio between the input to the output Signals in S domain with all the initial conditins are set equal to zero. Knowing the T.F. the output of the system can be examined for different input signals.



For the impulse function
 $x(t) = \delta(t) \Rightarrow X(s) = 1$

$$T.F. = H(s) = \frac{Y(s)}{X(s)},$$

$$H(s) = \frac{b_n s^n + \dots + b_1 s + b_0}{a_m s^m + \dots + a_1 s + a_0}$$

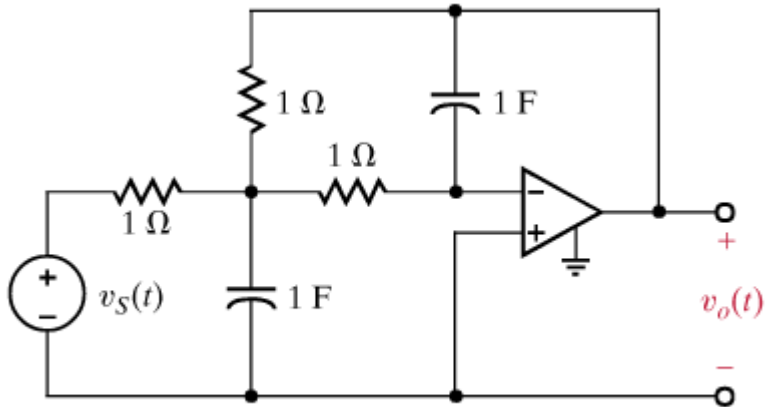
H(s) can also be interpreted as the Laplace transform of the output when the input is an impulse and all initial conditions are zero

The inverse transform of H(s) is also called the impulse response of the system

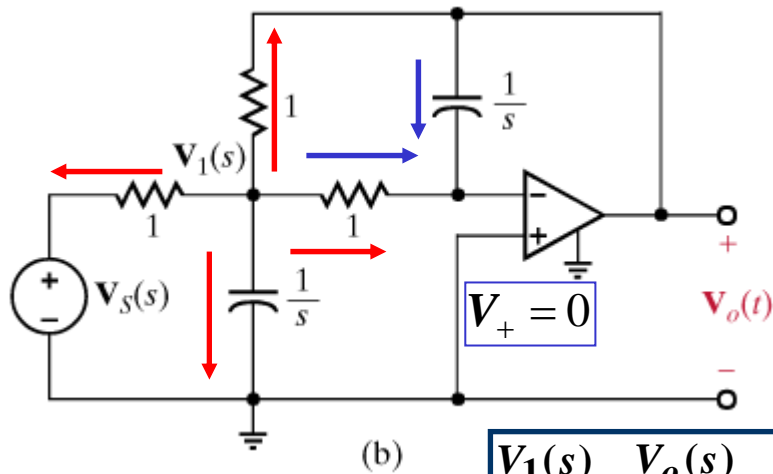
If the impulse response is known then one can determine the response of the system to ANY other input

EXAMPLE

Determine the transfer function, the type of damping and the unit step response



Transform the circuit to the Laplace domain. All initial conditions set to zero



$$V_1(s) = -sV_o(s) \Leftarrow$$

$$\frac{V_1(s)}{1} + \frac{V_o(s)}{1/s} = 0$$

$$\frac{V_1(s) - V_S(s)}{1} + \frac{V_1(s)}{1/s} + \frac{V_1(s)}{1} + \frac{V_1(s) - V_o(s)}{1} = 0$$

$$\frac{V_o(s)}{V_S(s)} = \frac{\frac{1}{32}}{s^2 + \frac{1}{2}s + \frac{1}{16}}$$

$$\text{Unit step response} \Rightarrow V_S(s) = \frac{1}{s}$$

$$V_o(s) = \frac{(1/32)}{s\left(s + \frac{1}{4}\right)^2}$$

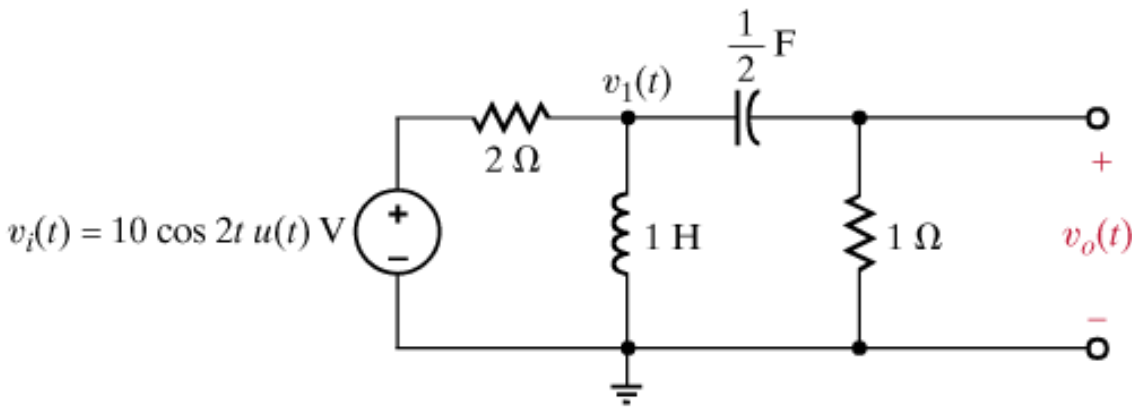
$$= \frac{K_o}{s} + \frac{K_{11}}{s + 0.25} + \frac{K_{12}}{(s + 0.25)^2}$$

$$K_o = sV_o(s) \big|_{s=0} = 0.5$$

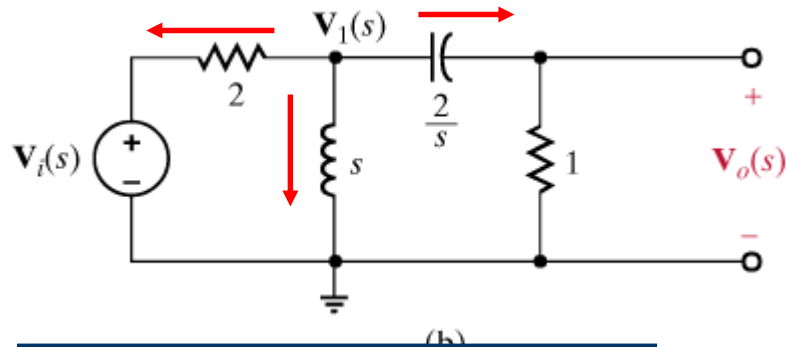
$$K_{12} = (s + 0.25)^2 V_o(s) \big|_{s=-0.25} = 0.125$$

$$K_{11} = \frac{d[s^2 V_o(s)]}{ds} \bigg|_{s=-0.25} = -0.5$$

$$v_o(t) = \left(0.5 - (0.125t + 0.5)e^{-0.25t}\right)u(t)$$

EXAMPLE**Determine the steady state response**

Transform the circuit to the Laplace domain.
Assume all initial conditions are zero



$$\text{KCL@}V_1: \frac{V_1 - V_i}{2} + \frac{V_1}{2} + \frac{V_1}{\frac{2}{s} + 1} = 0$$

$$\text{Voltage divider: } V_o = \frac{1}{\frac{2}{s} + 1} V_1$$

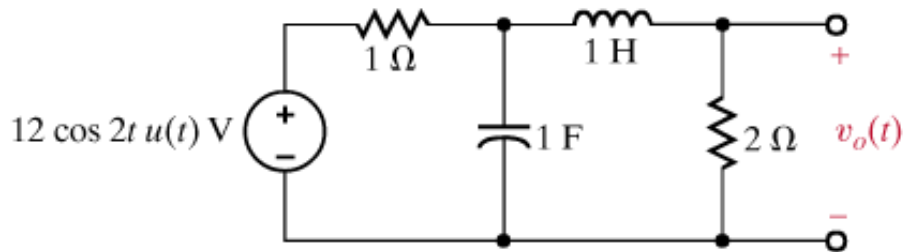
$$\text{If } x(t) = X_M \cos(\omega_o t + \theta) u(t)$$

$$y_{ss}(t) = |X_M| |H(j\omega_o)| \cos(\omega_o t + \angle H(j\omega_o) + \theta)$$

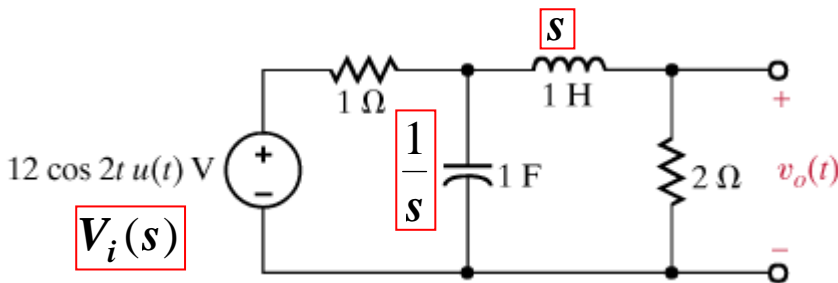
$$V_o(s) = \frac{s^2}{3s^2 + 4s + 4} V_i(s) \Rightarrow H(s) = \frac{s^2}{3s^2 + 4s + 4}$$

$$H(j2) = \frac{(j2)^2}{3(j2)^2 + 4(j2) + 4} = 0.354 \angle 45^\circ$$

$$\therefore y_s(t) = 3.54 \cos(2t + 45^\circ) \text{ V}$$

EXAMPLE**Determine $v_{oss}(t), t > 0$** 

**Transform circuit to Laplace domain.
Assume all initial conditions are zero**



$$V_{OC}(s) = \frac{\frac{1}{s}}{1 + \frac{1}{s}} V_i(s) = \frac{1}{s+1} V_i(s)$$

$$Z_{Th}(s) = s + \parallel 1, \frac{1}{s} \parallel = s + \frac{1}{s+1} = \frac{s^2 + s + 1}{s+1}$$

If $x(t) = X_M \cos(\omega_o t + \theta)u(t)$

$y_{ss}(t) = |X_M| |H(j\omega_o)| \cos(\omega_o t + \angle H(j\omega_o) + \theta)$

$$V_o(s) = \frac{2}{2 + Z_{Th}(s)} V_{OC}(s)$$

$$V_o(s) = \frac{2}{2 + \frac{s^2 + s + 1}{s+1}} \times \frac{1}{s+1} V_i(s)$$

$$V_o(s) = \frac{2}{s^2 + 3s + 3} V_i(s)$$

$$H(j2) = \frac{2}{-4 + 6j + 3} = \frac{2}{-1 + 6j} = \frac{2}{6.08 \angle 99.46^\circ}$$

$$v_{oss}(t) = 12 \times \frac{2}{6.08} \cos(2t - 99.46^\circ)$$